Level Set Framework, Signed Distance Function, and Various Tools

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Given a closed hypersurface, Γ, our goal is to evolve and track this hypersurface over time, $\Gamma(t)$, as it moves in the normal direction according to a given function, F. We do so by embedding Γ as the zero level set of a one dimension higher surface, ϕ. To be more specific about how this embedding works, we need some notation.

Notation:

At any specific time, t, we will denote

$$\begin{aligned} \Omega^+ &= \text{region outside } \Gamma(t). \\ \partial\Omega &= \Gamma(t). \\ \Omega^- &= \text{region inside } \Gamma(t). \end{aligned}$$

Define ϕ

We start off by defining ϕ at time $t = 0$ as,

Initialize ϕ

$$\phi(\vec{x}, t = 0) = \pm d, \qquad \vec{x} \in \mathbb{R}^n$$

We will discuss this d and the consequences of using it in then next section.

Level Set Equation

For each $\vec{x} \in \partial\Omega$, we want $\phi(\vec{x}(t), t) = 0$. Thus we have the identity

$$\frac{\partial}{\partial t}\phi(\vec{x}(t), t) = 0$$

so by chain rule, we get

$$\phi_t + \nabla\phi \cdot \vec{x}_t = 0$$

which by remembering that for each $\vec{x} \in \partial\Omega$, $\vec{x}_t \cdot \vec{n} = F(\vec{x})$. We get that

$$\nabla\phi \cdot \vec{x}_t = F|\nabla\phi|$$

so, we have the level set equation which when coupled with the initial location of Γ gives us an initial value pde.

Level Set Method

Initial Value Problem

$$\phi_t + F|\nabla\phi| = 0$$

$$\Gamma = \{\vec{x} \mid \phi(\vec{x}, t = 0) = 0\}$$

F can be a function of local, global or independent properties to ϕ. The real innovation in applications of the level set method is in the specific F used to give different effects to the front Γ.

Initialize ϕ

We defined our initial surface ϕ to be the Euclidean distance from $\vec{x}$ to Γ.

Distance Function

$$\phi(\vec{x}, t = 0) = \pm d, \qquad \vec{x} \in \mathbb{R}^n$$

Signed Distance Function

Formally, we define the distance from a point $\vec{x}$ to a set $\partial\Omega$

Distance Function

$$d(\vec{x}) = \min_{\vec{x}_C \in \partial\Omega} (|\vec{x} - \vec{x}_C|)$$

then our ϕ takes on the distance from the boundary with a sign depending on being inside or outside the region,

Signed Distance Function

$$\phi(\vec{x}) = \begin{cases} -d(\vec{x}) & \vec{x} \in \Omega^- \\ 0 & \vec{x} \in \partial\Omega \\ d(\vec{x}) & \vec{x} \in \Omega^+ \end{cases}$$

Why a Signed Distance Function?

- There are a number of properties of this specific definition of ϕ which help with the numerics of solving this level set problem.
- Any function which is negative inside and positive outside the initial front would work, but the signed distance function gives many nice properties. We will enumerate some in the slides to come.
- The Fast Marching Method is an efficient method for creating the signed distance function for any arbitrary initial closed fronts.

Properties of Signed Distance Function

Main Property

$$|\nabla\phi| = 1$$

- This is true for all points except if it they are equidistant from at least 2 points on the interface. This set of points is often called a **skeleton** in the literature and is a zero measure set.
- Luckily the existence of this skeleton does not cause any issues with the numerics as it will be smoothed over time. Thus, the fact that the gradient is not technically defined on this skeleton will not affect our numerical solutions to the level set equation.

Properties of Signed Distance Function

More Properties

- The Unit Normal Vector

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\nabla \phi}{1} = \nabla \phi$$

- Mean Curvature

$$\kappa = \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right) = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = \Delta \phi$$

Properties of Signed Distance Function

Closest Point on Interface

Given $\vec{x} \in \mathbb{R}^n$, then the closest point on the boundary $\partial\Omega$ is

$$\vec{x}_c = \vec{x} - \phi(\vec{x})\vec{n}$$

Differentiability on $\partial\Omega$

ϕ is differentiable on $\partial\Omega$ almost everywhere.

Convexity

If Ω^- is a convex region, then ϕ is a convex function.

Properties of Signed Distance Function

Given ϕ_1 and ϕ_2 signed distance functions for regions Γ_1 and Γ_2 respectively

Combinations of Signed Distance Functions

- $\phi(\vec{x}) = \min(\phi_1(\vec{x}), \phi_2(\vec{x}))$ represents the distance function for the **union** $\Gamma_1^- \cup \Gamma_2^-$.
- $\phi(\vec{x}) = \max(\phi_1(\vec{x}), \phi_2(\vec{x}))$ represents the distance function for the **intersection** $\Gamma_1^- \cap \Gamma_2^-$.
- $\phi(\vec{x}) = \max(\phi_1(\vec{x}), -\phi_2(\vec{x}))$ represents the distance function for the **set subtraction** $\Gamma_1^- \backslash \Gamma_2^-$.

Examples of Signed Distance Functions

Circle

A **circle** of radius r centered at (a, b).

$$\phi(\vec{x}) = \sqrt{(x-a)^2 + (y-b)^2} - r$$

Sphere

A **sphere** of radius r centered at a, b, c.

$$\phi(\vec{x}) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} - r$$

Examples of Signed Distance Functions

Ellipse

An **ellipse** centered at (a, b).

$$\phi(\vec{x}) = \sqrt{\frac{(x-a)^2}{A} + \frac{(y-b)^2}{B}} - 1$$

Square

A **square** of width w centered at (a, b).

$$\phi(\vec{x}) = \begin{cases} \max\left(|x-a| - w, |y-b| - w\right), & \vec{x} \in \Omega^+ \\ \min\left(|x-a| - w, |y-b| - w\right), & \vec{x} \in \Omega^- \end{cases}$$

Characteristic function χ.

A nice function for integrals is the characteristic function, sometimes called the indicator function.

Characteristic function of interior regions and exterior regions

$$\chi^{-}(\vec{x}) = \begin{cases} 1, & \text{if } \phi(\vec{x}) \leq 0 \\ 0, & \text{if } \phi(\vec{x}) > 0 \end{cases}$$

$$\chi^{+}(\vec{x}) = \begin{cases} 0, & \text{if } \phi(\vec{x}) \leq 0 \\ 1, & \text{if } \phi(\vec{x}) > 0 \end{cases}$$

Volume Integrals

The volume integral (area, length) of a function f on a region Ω^- is

Volume integral

$$V(\Omega^-) = \int_{\mathbb{R}^n} f(\vec{x})\chi^-(\vec{x})d\vec{x}$$

Heaviside Function

We can define the heaviside function to do the same job as the characteristic function as a function of ϕ.

Heaviside Function

$$H(\phi) = \begin{cases} 0, & \text{if } \phi \leq 0 \\ 1, & \text{if } \phi > 0 \end{cases}$$

which allows us to define the volume integrals

$$V(\Omega^-) = \int_{\mathbb{R}^n} f(\vec{x})[1 - H(\phi(\vec{x})]d\vec{x}$$

$$V(\Omega^+) = \int_{\mathbb{R}^n} f(\vec{x})H(\phi(\vec{x})d\vec{x}$$

Dirac Delta Function

We define the Dirac delta function to be the directional derivative of the heaviside function, H in the normal direction $\vec{n}$.

$$\begin{aligned}\hat{\delta}(x) &= \nabla H(\phi(x)) \cdot \vec{n} \\ &= H'(\phi(x))\, \nabla\phi(x) \cdot \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \\ &= H'(\phi(x))\, |\nabla\phi(x)| \\ &= \delta(\phi(x))\, |\nabla\phi(x)|\end{aligned}$$

where

$$\delta(\phi) = H'(\phi)$$

is the 1-D version of the Dirac delta function which evaluates to 1 where $\phi = 0$ and 0 everywhere else.

Surface Integrals

The surface integral of a function f over a boundary $\partial\Omega$ is defined to be

Surface integral

$$\int_{\mathbb{R}^n} f(\vec{x})\hat{\delta}(\vec{x})d\vec{x}$$

Smeared Heaviside Function

When doing these numerically, we use a smeared-out heaviside and delta function to make the calculations.

Smeared-out Heaviside and Dirac delta functions

$$H(\phi) = \begin{cases} 0, & \phi < -\epsilon \\ \frac{1}{2} + \frac{\phi}{2\epsilon} + \frac{1}{2\pi}\sin\left(\frac{\pi\phi}{\epsilon}\right), & \epsilon \leq \phi \leq \epsilon \\ 1, & \epsilon < \phi \end{cases}$$

$$\delta(\phi) = \begin{cases} 0, & \phi < -\epsilon \\ \frac{1}{2\epsilon} + \frac{1}{2\epsilon}\cos\left(\frac{\pi\phi}{\epsilon}\right), & -\epsilon \leq \phi \leq \epsilon \\ 0, & \epsilon < \phi \end{cases}$$

where ϵ is a tunable parameter for the smearing bandwidth. Osher recommends to use $\epsilon = 1.5\Delta x$.

Smeared Heaviside Function

The smeared-out heaviside and delta function lead to first order accurate, $\mathcal{O}(\Delta x)$, calculations of the surface and volume integrals. For the signed distance function, $|\nabla \phi| = 1$, so we get the simplified versions

Simplified Volume and Surface Integrals for signed distance function

$$\int_{\mathbb{R}^n} f(\vec{x})\delta(\phi(\vec{x}))d\vec{x} = \text{Surface integral of f on } \partial\Omega$$

$$\int_{R^n} f(\vec{x})H(\phi(\vec{x}))d\vec{x} = \text{Volume integral of f on } \Omega^-$$

Note that if $f = 1$, then this yields the surface area or volume of $\partial\Omega$ or Ω^- respectively.

Any Questions?