Introduction to Level Set Methods for Front Evolution

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Suppose we have an interface which evolves according to some speed function $F$ in the normal direction.

Figure: Curve propagating with speed $F$ in normal direction
The traditional way to simulate this motion has been to parameterize the curve and then place markers along the parameterization. We then evolve those markers and interpolate the curve as they move.
Problems with Markers

This method works remarkably well when the interface is well-behaved. However, what happens when the curve pinches?

Pinching of interface

Or when it splits?

Splitting interface
In 1989, J.A. Sethian and Stanley Osher presented a new technique to deal with these types of topology changes which they called the level set method.
What is a Level Set?

For a given surface $f(\vec{x})$, we say that the set of points $\vec{x}$ such that $f(\vec{x}) = c$ for $c \in \mathbb{R}$ is the $c$ level set, denoted:

$$f^{-1}(c) = \{\vec{x} \mid f(\vec{x}) = c\}$$

**Figure:** Surface and Contour of Zero Level Set $f^{-1}(0)$. 
Given an initial closed \( n - 1 \) dimensional hyper-surface, \( \Gamma \), we want to evolve it over time, \( \Gamma(t) \), as it propagates along its normal direction according to a speed function \( F \).
Derivation of Level Set Evolution Equation, cont’d

We thus embed $\Gamma$ as the zero level set of a higher dimensional function $\phi$. Let $\phi(\vec{x}, t = 0)$, for $\vec{x} \in \mathbb{R}^n$, be defined by

$$\phi(\vec{x}, t = 0) = \pm d$$

where $d$ is the distance from $\vec{x}$ to $\Gamma(t = 0)$, and the plus sign is chosen for $\vec{x}$ outside the initial hyper-surface and the minus sign for $\vec{x}$ inside the initial hyper-surface.

Figure: Unit Circle Embedded in Three Space
Thus we have a function $\phi(\vec{x}, t = 0) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\Gamma(t = 0) = \{ \vec{x} \mid \phi(\vec{x}, t = 0) = 0 \}$$

and we want to produce an equation for evolving the function $\phi(\vec{x}, t)$ which will always contain $\Gamma(t)$ as the level set $\phi = 0$. 
In addition, we want the speed of motion in the normal direction to be $F$ or in other words, for each $\vec{x}(t) \in \Gamma(t)$,

$$\vec{x}_t \cdot \vec{n} = F(\vec{x}(t))$$

where $\vec{n}$ is normal to the front at $\vec{x}(t)$ and $F(\vec{x}(t))$ is the speed function for that point.
Derivation of Level Set Evolution Equation, cont’d

We want the zero level set of $\phi$ to match our propagating hyper-surface, so we have the condition

$$\phi(\vec{x}(t), t) = 0, \forall \vec{x}(t) \in \Gamma(t)$$

Thus

$$\frac{\partial}{\partial t} \phi(\vec{x}(t), t) = 0$$

so by chain rule,

$$\phi_t + \nabla \phi(\vec{x}(t), t) \cdot \vec{x}_t(t) = 0$$

and since $\vec{n} = \nabla \phi / |\nabla \phi|$, we have that $\nabla \phi = \vec{n} |\nabla \phi|$, so

$$\phi_t + \vec{n} \cdot \vec{x}_t(t) |\nabla \phi| = 0$$

or in other words

$$\phi_t + F |\nabla \phi| = 0$$

which is our level set evolution equation.
The Level Set Method

Given $\Gamma$, an initial closed hypersurface and a forcing function $F$, solve the pde

$$\phi_t + F|\nabla \phi| = 0$$

with initial condition

$$\{ \vec{x} \mid \phi(\vec{x}, t = 0) = 0 \} = \Gamma$$

and at any given time $\Gamma(t) = \{ \vec{x} \mid \phi(\vec{x}, t) = 0 \}$ is our evolved interface.
Benefits of Level Set Methods

- Handles topology changes easily.
- Scales for any dimension 2D, 3D or even more if needed.
- Much research has been done to date in Computational Physics to numerically solve this equation.
- By changing the forcing function $F$, you can get many different effects.
- Using the Fast Marching method, it is relatively easy to embed any initial shaped boundary into the higher dimensional surface.
Choosing F, the Speed Function

By choosing different F’s, we can achieve different effects from simple advection of the object to motion in the normal direction or by choosing fancy combinations of different properties, you can achieve edge detection or even simulate motion of fire flames. Choosing F becomes the key to achieving different effects and generates the bulk of the work in applications. F can depend on three properties of our front:

\[ F = F(L, G, I) \]

- \( L \) = Local properties like curvature and normal direction
- \( G \) = Global properties of the front which depend on the shape and position of the front
- \( I \) = Independent properties like fluid velocity passively transporting the front
Let

$$F = -\nabla \cdot \left( g(\nabla u_0) \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \right)$$

where $g$ is the edge detector,

$$g(\nabla u_0) = \frac{1}{1 + |J \ast \nabla u_0|^p}$$

and $p \geq 1$ and $J$ is a gaussian kernel with standard deviation $\sigma$. Then our level set equation becomes, after some simplification,

$$\phi_t = g(\nabla u_0) \kappa |\nabla \phi| + \nabla g(\nabla u_0) \cdot \nabla \phi$$

so we see an edge detection/curvature motion and a convection in direction of the gradient of the edge detector.
**Angiogram Example of Edge Detection**

![Figure: Finding the edge of a vein](image)

As described on [http://math.berkeley.edu/~sethian/2006/Applications/Medical_Imaging/artery.html](http://math.berkeley.edu/~sethian/2006/Applications/Medical_Imaging/artery.html) by J.A. Sethian.
Let

\[ F = -\beta \kappa = -\beta \nabla \cdot \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \]

then,

\[ \phi_t = \beta \kappa |\nabla \phi| \]

which becomes a nonlinear heat equation. This has the property that

- sharp boundaries are preserved
- smoothing takes place inside a region but not across region boundaries
Example with a picture of the retina

Figure: Smoothing a retina
Example noise removal using $F_{min/max}$

Figure: Letters with 25% noise
Example noise removal using $F_{\text{min}}/\text{max}$

Figure: Recovered Letters
Setup for Inverse Scattering

Let $A$ be a differential equation and $g$ initial data of problem such that

$$A(u) = g$$

we want to solve for the $u$ which gives us our observed data $g$. Let

$$W(u) = \frac{1}{2} \| A(u) - g \|^2$$

then we want to find a solution $u$ which minimizes $W$. Fadil Santosa recast this problem in level set notation and came up with the evolutionary equation which solves this problem

$$\phi_t + \left[ J(u)^T (A(u) - g) \right] |\nabla \phi| = 0$$

where $J(u)^T$ is the Jacobain of $A(u)$ at $u$. 
From Dorn & Lesselier 2006,
Image Recognition from Sethian, ”Level Set Methods and Fast Marching Methods”, 1996

Combine the inverse image problem with neural networks and you can get image identification and shape recognition.

Figure: Perturbations of numeral under flow rules

Compare perturbations with NIST database to identify the letter.
Stereo Problem

The goal is to take many images of an object from different directions and angles and then reconstruct the surface in a 3D model.

Figure: Different views of the same point M
The language of stereo problems is involved but it reduces to starting with an initial surface $S$ and flowing in the normal direction

$$S_t = \beta N$$

where $\beta$ is a functional which measures matching between projection images.

These methods are completely treated in *Complete Dense StereoVision using Level Set Methods* by Olivier Faugeras and Renaud Keriven, 1998.
Techniques in Physics

- compressible flow
- solid-fluid coupling
- incompressible flow (smoke plumes)
- free surfaces (water splashing)
- flame propagation
- movie animation
Other Applications in Animation

- melting of Terminatrix in Terminator 3
- liquids, beer, water, mud in Shrek

Figure: Terminatrix, and Shrek
Other Applications

- Optimal Path/ Robotics Navigation
- Etching for Semiconductor Manufacturing
- Grid Generation
- Minimal Surfaces (Curvature = 0)
- Geometry
Any Questions?